Velocity addition in Special Relativity and in Newtonian Mechanics are isomorphic

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Abstract

In the one dimensional case, velocity addition in Special Relativity and in Newtonian Mechanics, respectively, are each a commutative group operation, and the two groups are *isomorphic*. There are *infinitely* many such isomorphisms, each indexed by one positive real parameter.

1. Velocity addition in Special Relativity

Let $c > 0$ be the velocity of light in vacuum. Then, as is well known, Angel, in the case of uniform motion along a straight line, the special relativistic addition of velocities is given by

(SR) $$u * v \ = \ (u+v)/(1+uv/c^2), \quad u, v \in (-c, c)$$

thus the binary operation $*$ acts according to

$$* : (-c, c) \times (-c, c) \longrightarrow (-c, c)$$

It follows immediately that

1) $*$ is associative and commutative

2) $u * v * w \; = \; (u + v + w + uvw/c^2)/(1 + (uv + uw + vw)/c^2)$

for $u, v, w \in (-c, c)$

3) $u * 0 \; = \; 0 * u \; = \; u, \quad u \in (-c, c)$

4) $u * (-u) \; = \; (-u) * u \; = \; 0, \quad u \in (-c, c)$

5) $\partial/\partial u(u * v) \; = \; (1 - v^2/c^2)/(1 + uv/c^2)^2 > 0, \quad u, v \in (-c, c)$

6) $\lim_{u,v \to c} \; u * v \; = \; c, \qquad \lim_{u,v \to -c} \; u * v \; = \; -c$

Therefore

7) ($(-c, c), *$) is a commutative group with the neutral element 0, while $-u$ is the inverse element of $u \in (-c, c)$

2. Velocity addition in Newtonian Mechanics

As is well known, in the case of uniform motion along a straight line, the addition of velocities in Newtonian Mechanics is given by

(NM) $\qquad x + y, \quad x, y \in \mathbb{R}$

thus it is described by the usual additive group $(\mathbb{R}, +)$ of the real numbers, a group which is of course commutative, with the neutral element 0, while $-x$ is the inverse element of $x \in \mathbb{R}$.

3. Isomorphisms of the two groups

8) ($(-c, c), *$) and $(\mathbb{R}, +)$ are isomorphic groups through the mappings

8.1) $\quad \alpha : (-c, c) \longrightarrow \mathbb{R}$, where

$$\alpha(u) \; = \; k \ln((c + u)/(c - u)), \quad u \in (-c, c)$$

and

8.2) $\beta : \mathbb{R} \longrightarrow (-c, c)$, where

$\beta(x) \ = \ c(e^{x/k} - 1)/(e^{x/k} + 1), \quad x \in \mathbb{R}$

with

8.3) $k \ = \ c^2 \alpha'(0) > 0$

Proof of 8)

Let us first find α. According to the standard definition of group homomorphism, we have

$$\alpha \text{ group homomorphism } \Leftrightarrow \alpha(u * v) = \alpha(u) + \alpha(v), \ \ u, v \in (-c, c)$$

Thus it follows that

$$\alpha(u * v) - \alpha(u) \ = \ \alpha(v), \quad u, v \in (-c, c)$$

and since the right hand term does not depend on u, we conclude that neither does the left hand term. Consequently, assuming that α has a derivative on its domain of definition $(-c, c)$, we obtain

$$d/du \ (\alpha(u * v) - \alpha(u)) \ = \ 0, \quad u, v \in (-c, c)$$

or in view of (SR) and 5), the relation follows

$$\alpha'((u + v)/(1 + uv/c^2))((1 - v^2/c^2)/(1 + uv/c^2)) \ = \ \alpha'(u)$$

for $u, v \in (-c, c)$

Taking now $u = 0$, one obtains

$\alpha'(v)(1 - v^2/c^2) \ = \ \alpha'(0), \quad v \in (-c, c)$

or

$$\alpha'(v) \ = \ c^2\alpha'(0)/(c^2 - v^2), \quad v \in (-c, c)$$

Thus, since $\alpha(0) \ = \ 0$ results form the fact that α is assumed to be a group homomorphism, one obtains

$$\alpha(u) \ = \ \alpha(0) + c^2\alpha'(0) \int_0^u dv/(c^2 - v^2) \ = \ c^2\alpha'(0) \int_0^u dv/(c^2 - v^2) \ =$$

$$= \ c^2\alpha'(0) \ln((c+u)/(c-u)), \quad u \in (-c, c)$$

in other words, 8.1) and 8.3). And since obviously the resulting α in 8.1) is a *bijective* mapping, it follows that it is not only a group homomorphism, but also a group isomorphism. In this way, its inverse mapping $\beta = \alpha^{-1}$ exists and it is also a group isomorphism. Finally, a simple computation based on 8.1) will then give 8.2).

4. Note

The special relativistic addition $*$ of velocities in (SR) is in fact well defined not only for pairs of velocities

$$(u, v) \in (-c, c) \times (-c, c)$$

but also for the *larger* set of pairs of velocities

$$(u, v) \in [-c, c] \times [-c, c], \quad uv \neq -c^2$$

This corresponds to the fact that in Special Relativity the velocity c of light in vacuum is supposed to be attainable.

On the other hand, the Newtonian addition $+$ of velocities (NM) does of course only make sense physically for

$$(x, y) \in \mathbb{R} \times \mathbb{R}$$

since infinite velocities are not supposed to be attainable physically.

As for the group isomorphisms α and β, they only generate mappings between pairs of velocities in

$$(-c, c) \times (-c, c) \xrightarrow{\alpha} \mathbb{R} \times \mathbb{R}$$

and

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\beta} (-c, c) \times (-c, c)$$

thus they do *not* cover the cases of addition $u * v$ of special relativistic velocities $u = -c$ and $v < c$, or $-c < u$, and $v = c$.

Consequently, in spite of the group isomorphisms α and β, there is an *essential* difference between the addition of velocities in Special Relativity, and on the other hand, Newtonian Mechanics. Indeed, in the latter case, the addition $+$ is defined on the *open* set $\mathbb{R} \times \mathbb{R}$, while in the former case the addition $*$ is defined on the set

$$\{\ (u, v) \mid -c \leq u, v \leq c, \quad uv \neq -c^2\ \}$$

which is *neither open, nor closed*.

5. The uniqueness of the velocity addition in Special Relativity

In Benz, it has recently be shown that under very general and mild conditions, the formula (SR) is *uniquely* determined, even if one starts with motions not along a straight line, but in arbitrary, thus possibly infinite dimensional pre-Hilbert spaces as well.

Reference

1. Angel, Roger B : Relativity, The Theory and its Philosophy. Pergamon, New York, 1980

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